Cryptography ECE5632 - Spring 2025

Lecture 7A

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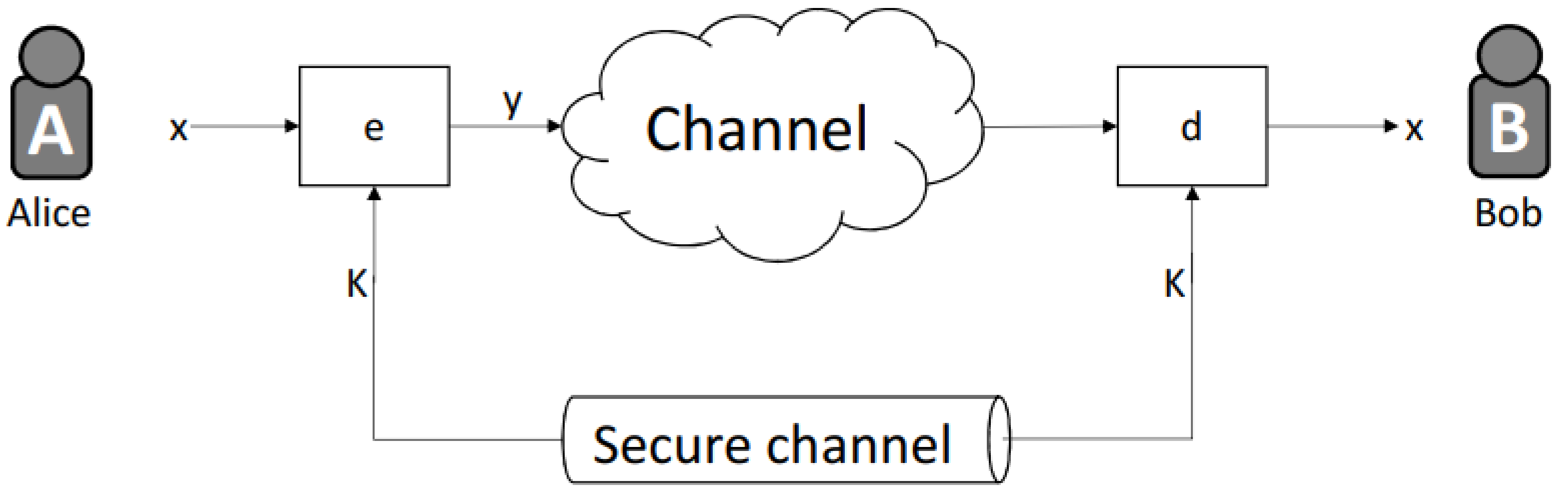
Lecture Topic

Introduction to Public-Key Cryptography (PKC)

Symmetric Cryptography Revisited

➢ Principle of Symmetric-Key encryption:

- The **same secret key** ***K*** is used for encryption and decryption
- Encryption and Decryption are very similar (or even identical) functions



X: plaintext
Y: ciphertext
K: key

❑ We have some problems; key distribution, number of keys, etc.

Symmetric Cryptography Revisited

➢ Symmetric Cryptography: Analogy

Safe with a strong lock, only Alice and Bob have a copy of the key

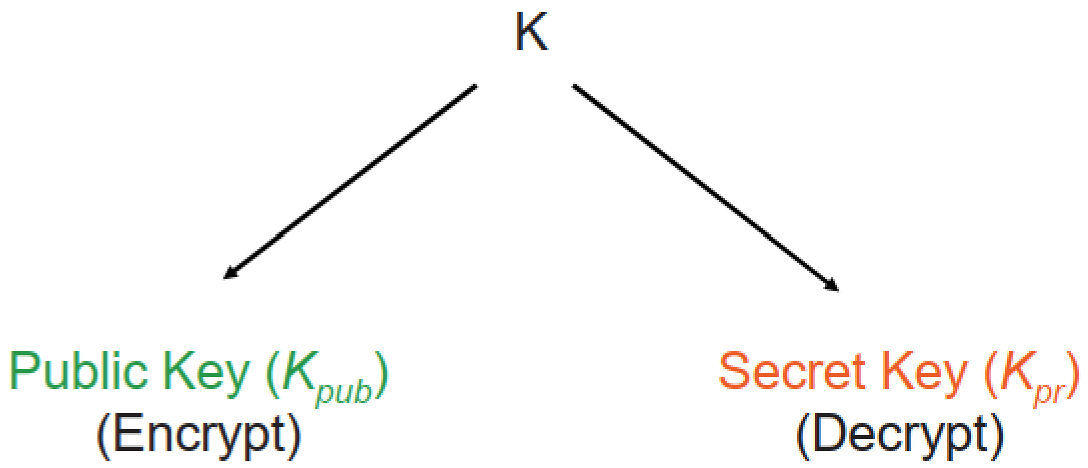
- Alice encrypts : locks message in the safe with her key
- Bob decrypts : uses his copy of the key to open the safe

Principles of Asymmetric Cryptography

➢ **Idea behind Asymmetric Cryptography**

Asymmetric (Public-Key) Cryptography

- ➢ Principle: “Split up” the key



- ✓ During the key generation, a key pair K_{pub} and K_{pr} is computed

Asymmetric (Public-Key) Cryptography

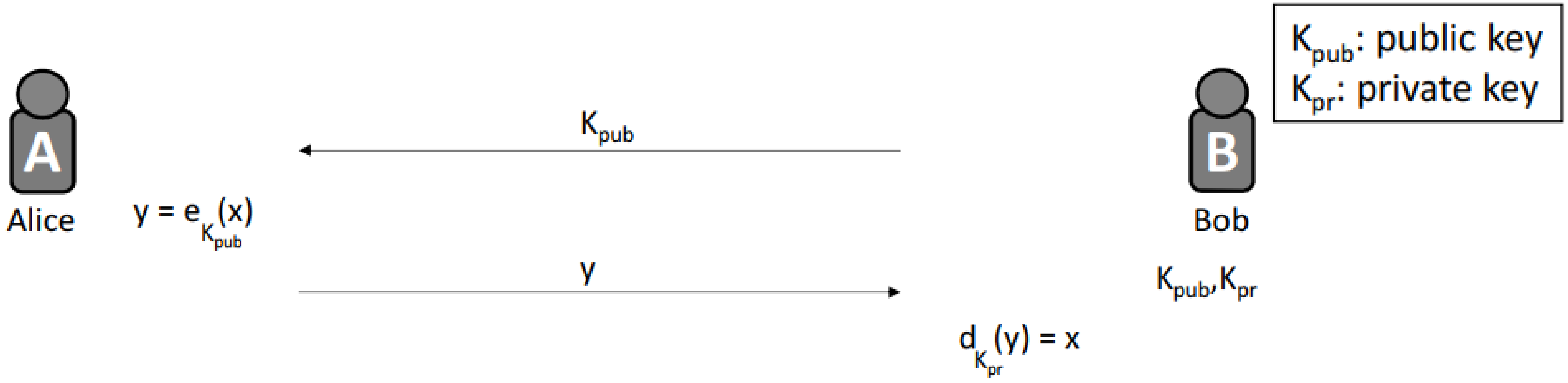
➢ Asymmetric Cryptography: Analogy

Safe with public lock and private lock:

- Alice deposits (encrypts) a message with the - *not secret* - public key K_{pub}
- Only Bob has the - *secret* - private key K_{pr} to retrieve (decrypt) the message

Asymmetric (Public-Key) Cryptography

➢ Basic protocol for Public-Key encryption:



✓ To study e() and d() .. **more Math is needed…**

Essential Number Theory for PKC

Euclidean Algorithm (EA)

➢ Compute the **greatest common divisor** ***gcd (*r_0*,* r_1*)*** of two integers r_0 and r_1

❖ gcd is **easy for small numbers**:

1. factor r_0 and r_1
2. gcd = highest common factor

Example 1 : $r_0 = 84, r_1 = 30$

$$r_0 = 84 = 2 \cdot 2 \cdot 3 \cdot 7$$
$$r_1 = 30 = 2 \cdot 3 \cdot 5$$

→ The gcd is the product of all common prime factors:

$2 \cdot 3 = 6 = gcd\ (30,84)$

❖ **But:** Factoring is complicated (and often infeasible) for large numbers

Euclidean Algorithm (EA)

Such method doesn't work with large numbers, i.e., the case of PKC. **We need the EA.**

e.g., 3^{2000}

Efficient
(faster, less complex)

Basic idea: $\gcd(r_0, r_1) = \gcd(r_0 \bmod r_1, r_1)$

$= \gcd(r_1, r_0 \bmod r_1)$

. . . we simply reduce the problem.

Example 1 : $r_0 = 84, r_1 = 30$

gcd(84, 30) = gcd(84 mod 30, 30) = gcd(**24**, 30)

= gcd(30 mod 24, 24) = gcd(**6**, 24) = 6

= gcd(24 mod 6, 6) = gcd(**0**, 6)

Terminate once a zero remainder is reached; gcd is the last remainder.

Same e.g., (illustrated)

$\overset{r_0}{84} = 2 \cdot \overset{r_1}{30} + \overset{r_2}{24}$

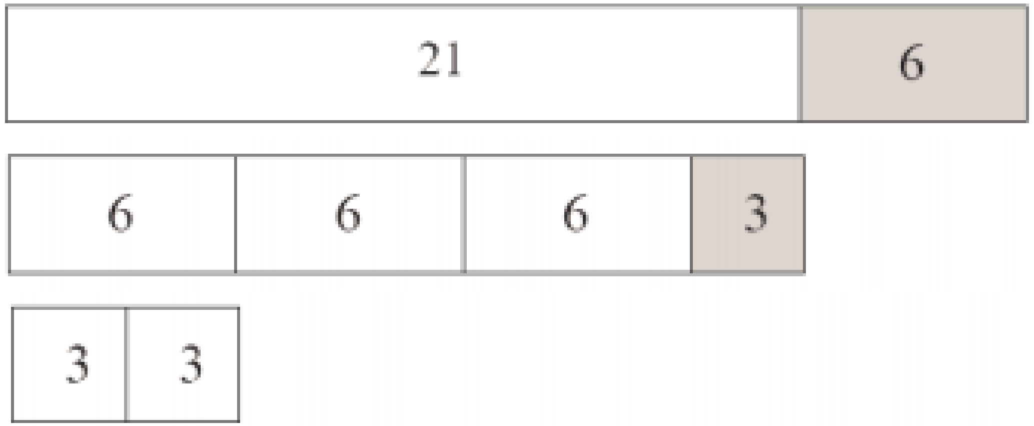
$30 = 1 \cdot 24 + \overset{r_3}{6}$ → gcd (84, 30) = 6

$24 = 4 \cdot 6 + 0$ . . . zero remainder reached.

Euclidean Algorithm (EA)

Example 2 : $\mathbf{r_0 = 27}$, $\mathbf{r_1 = 21}$

gcd (r_0, r_1) for r_0 = 27 and r_1 = 21



$\gcd(27, 21) = \gcd(1 \cdot 21 + 6, 21) = \gcd(21, 6)$

$\gcd(21, 6) = \gcd(3 \cdot 6 + 3, 6) = \gcd(6, 3)$

$\gcd(6, 3) = \gcd(2 \cdot 3 + 0, 3) = \gcd(3, 0) = 3$

- Note: very efficient method even for long numbers:
  The complexity grows **linearly** with the number of bits

Euclidean Algorithm (EA)

Example 3 : $\mathbf{r_0 = 973\ ,r_1 = 301}$

$\gcd(\overset{r_0}{973}, \overset{r_1}{301})$

$\overset{r_0}{973} = 3 \cdot \overset{r_1}{301} + \overset{r_2}{70}$

$301 = 4 \cdot 70 + \overset{r_3}{21}$

$70 = 3 \cdot 21 + \overset{r_4}{7}$ → $\gcd(973, 301) = 7$

$21 = 3 \cdot 7 + 0$ . . . zero remainder reached.

Pretty simple…

Extended Euclidean Algorithm (EEA)

Goal: rewrite $\gcd(r_0, r_1) = s \cdot r_0 + t \cdot r_1$

Why and **How**?

Why: To compute <u>modular inverses of large numbers</u>.

How: Using regular EA for the LHS & the extension

Extended Euclidean Algorithm (EEA)

- Extend the Euclidean algorithm to **find modular inverse** of r_1 mod r_0
- EEA computes *s*,*t*, and the gcd : $\gcd(r_0, r_1) = s \cdot r_0 + t \cdot r_1$
- Take the relation **mod r_0**

$$s \cdot r_0 + t \cdot r_1 = 1$$

$$s \cdot 0 + t \cdot r_1 \equiv 1 \bmod r_0$$

$$r_1 \cdot t \equiv 1 \bmod r_0$$

→ Compare with the definition of modular inverse**: *t* is the inverse of r_1 mod r_0**

- Note that *gcd* $(r_0, r_1) = 1$ in order for the inverse to exist

- **Recursive formulae** to calculate *s* and *t* in each step

  → „magic table" for *r*, *s*, *t* and a quotient *q* to derive the inverse with pen and paper

Extended Euclidean Algorithm

- How to compute the modular inverse using the Extended Euclidean Algorithm:

i	$q_i = \left\lfloor \frac{r_{i-2}}{r_{i-1}} \right\rfloor$	$s_i = s_{i-2} - q_{i-1} \cdot s_{i-1}$	$t_i = t_{i-2} - q_{i-1} \cdot t_{i-1}$	$r_i = r_{i-2} - q_{i-1} \cdot r_{i-1}$
0		$s_0 = 1$	$t_0 = 0$	r_0
1		$s_1 = 0$	$t_1 = 1$	r_1
2	$q_1 = \left\lfloor \frac{r_0}{r_1} \right\rfloor$	$s_2 = s_0 - q_1 \cdot s_1$	$t_2 = t_0 - q_1 \cdot t_1$	$r_2 = r_0 - q_1 \cdot r_1$
3	$q_2 = \left\lfloor \frac{r_1}{r_2} \right\rfloor$	$s_3 = s_1 - q_2 \cdot s_2$	$t_3 = t_1 - q_2 \cdot t_2$	$r_3 = r_1 - q_2 \cdot r_2$
⋮	⋮	⋮	⋮	⋮

For initialization (steps $i \in \{0,1\}$, cell values are predetermined as proven before.

For $i \geq 2$, compute the q_i, s_i, t_i, r_i columns.

For each iteration *i*, check:

If $r_i=1$ is reached, then **$\gcd(r_0,r_1) = r_i = 1$**. Then **mult. inverse of r_1 mod r_0 exists and equals t_i**. Stop.

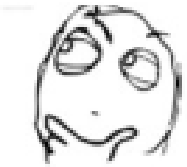
Else, if $r_i=0$ is reached, **then $\gcd(r_0,r_1) = r_{i-1}$**. Then **mult. inverse of r_1 mod r_0 doesn't exist**. Stop.

Extended Euclidean Algorithm (EEA)

$\gcd(r_0, r_1)$	$r_0 = q_1 \cdot r_1 + r_2$	$r_2 = s_2 \cdot r_0 + t_2 \cdot r_1$
$\gcd(r_1, r_2)$	$r_1 = q_2 \cdot r_2 + r_3$	$r_3 = s_3 \cdot r_0 + t_3 \cdot r_1$
	$\vdots$	$\vdots$
$\gcd(r_{l-2}, r_{l-1})$	$r_{l-2} = q_{l-1} \cdot r_{l-1} + r_l$	$r_l = \overset{s}{s_l} \cdot r_0 + \overset{t}{t_l} \cdot r_1 = \gcd(r_0, r_1)$
$\gcd(r_{l-1}, r_l)$	$r_{l-1} = q_l \cdot r_l + 0$	

r_l is the last remainder, i.e., the result of $\gcd(r_0, r_1)$.

How does that lead to modulo inverse?



To compute $a^{-1} \bmod n$:

$\gcd(n, a) = r_l = s \cdot n + t \cdot a = 1$ (condition for inverse existence)

Then $s \cdot 0 + t \cdot a \equiv 1 \bmod n$ (mod n for both sides)

$t \cdot a \equiv 1 \bmod n$

$t \equiv a^{-1} \bmod n$

Extended Euclidean Algorithm (EEA)

Example:

- Calculate the modular Inverse of 12 mod 67:
- From magic table follows $-5 \cdot 67 + 28 \cdot 12 = 1$
- Hence **28 is the inverse** of 12 mod 67.

i	q_{i-1}	r_i	s_i	t_i
2	5	7	1	-5
3	1	5	-1	6
4	1	2	2	-11
5	2	1	-5	**28**

- Check: $28 \cdot 12 = 336 \equiv 1 \bmod 67$ ✓

Multiplicative Inverse in GF(2^m)

Find the multiplicative inverse of x^3+x+1 in GF(2^4) with $P(x)=x^4+x+1$

Using EEA:

$x^4+x+1 = (x)(x^3+x+1) + (x^2+1)$ . . . $x^2+1 = (x^4+x+1) + (x)(x^3+x+1)$

$(x^3+x+1) = (x)(x^2+1) + 1$ . . . $1 = (x)(x^2+1) + (x^3+x+1)$

$= (x)(x^4+x+1) + (x^2+1)(x^3+x+1)$

$[(x)(x^4+x+1) + (x^2+1)(x^3+x+1)] \bmod P(x)$ → multiplicative inverse $= x^2+1$

Multiplicative Inverse in GF(2^m)

Example We are looking for the inverse of $A(x) = x^2$ in the finite field $GF(2^3)$ with $P(x) = x^3 + x + 1$. The initial values for the $t(x)$ polynomial are: $t_0(x) = 0$, $t_1(x) = 1$

Iteration	$r_{i-2}(x)$	$= [q_{i-1}(x)]\, r_{i-1}(x) + [r_i(x)]$	$t_i(x)$
2	$x^3 + x + 1$	$= [x]\, x^2 + [x + 1]$	$t_2 = t_0 - q_1 t_1 = 0 - x\,1 \equiv x$
3	x^2	$= [x]\,(x + 1) + [x]$	$t_3 = t_1 - q_2 t_2 = 1 - x\,(x) \equiv 1 + x^2$
4	$x + 1$	$= [1]\, x + [1]$	$t_4 = t_2 - q_3 t_3 = x - 1\,(1 + x^2)$
			$t_4 \equiv 1 + x + x^2$
5	x	$= [x]\, 1 + [0]$	Termination since $r_5 = 0$

$$A^{-1}(x) = t(x) = t_4(x) = x^2 + x + 1.$$

Multiplicative Inverse in GF(2^m)

Here is the check that $t(x)$ is in fact the inverse of x^2, where we use the properties that $x^3 \equiv x+1 \bmod P(x)$ and $x^4 \equiv x^2+x \bmod P(x)$:

$$\begin{aligned} t_4(x) \cdot x^2 &= x^4 + x^3 + x^2 \\ &\equiv (x^2+x) + (x+1) + x^2 \bmod P(x) \\ &\equiv 1 \bmod P(x) \end{aligned}$$

Euler's Phi Function

For PKC, it's important to know how many numbers in Z_m that are relatively prime to m.

Why and **how**?

Why: Will be clear later once we study actual PK cryptosystems.

How: Using Euler's Phi function simply counts these numbers.

Manually counting may work for small numbers.

e.g., manually count the numbers in Z_6 that are relatively prime to 6. → $\Phi(6) = 2$

For large numbers, we use Euler's Phi function.

Euler's Phi Function

- For PKC, it's important to know how many numbers in Z_m that are relatively prime to m.

Why and **how**?

Why: Will be clear later once we study actual PK cryptosystems.
How: Using Euler's Phi function simply counts these numbers.

- ***New problem,*** *important for public-key systems, e.g., RSA:*
  Given the set of the *m* integers {0, 1, 2, …, *m* -1},
- **How many** numbers in the set are **relatively prime to *m*** ?
  - **Answer:** **Euler's Phi function $\Phi(m)$**

Euler's Phi Function

- **Example** for the sets {0,1,2,3,4,5} (*m*=6), and {0,1,2,3,4} (*m*=5)

$$\gcd(0,6) = 6$$
$$\gcd(1,6) = 1 \leftarrow$$
$$\gcd(2,6) = 2$$
$$\gcd(3,6) = 3$$
$$\gcd(4,6) = 2$$
$$\gcd(5,6) = 1 \leftarrow$$

→ 1 and 5 relatively prime to *m*=6,

hence **$\Phi(6) = 2$**

$$\gcd(0,5) = 5$$
$$\gcd(1,5) = 1 \leftarrow$$
$$\gcd(2,5) = 1 \leftarrow$$
$$\gcd(3,5) = 1 \leftarrow$$
$$\gcd(4,5) = 1 \leftarrow$$

→ **$\Phi(5) = 4$**

- Testing one gcd per number in the set is **extremely slow for large *m*.**

Euler's Phi Function

- Manually counting may work for **small numbers**.

e.g., manually count the numbers in Z_6 that are relatively prime to 6. → $\Phi(6) = 2$

✓ For **large numbers**, we use Euler's Phi function.

Euler's Phi Function

- **If** canonical factorization of m known: $m = p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_n^{e_n}$
  (where p_i primes and e_i positive integers)
- **then** calculate Phi according to the relation $\Phi(m) = \prod_{i=1}^{n} (p_i^{e_i} - p_i^{e_i - 1})$

- Phi especially easy for $e_i = 1$, e.g., $m = p \cdot q \rightarrow \Phi(m) = (p-1) \cdot (q-1)$

- **Example** $m = 899 = 29 \cdot 31$:
  Φ(899) = (29-1) · (31-1) = 28 · 30 **= 840**

- **Note:** Finding $\Phi(m)$ is computationally easy **if factorization of *m* is known**
  (otherwise the calculation of $\Phi(m)$ becomes computationally infeasible for large numbers)

Euler’s Phi Function

How to compute $\Phi(m)$ for a large m?

Let m have the following factorization form:

$$m = p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_n^{e_n},$$

Where p_i are distinct prime numbers and e_i are positive integers, then

$$\Phi(m) = \prod_{i=1}^{n} (p_i^{e_i} - p_i^{e_i - 1}).$$

Euler's Phi Function

e.g. 1) compute $\Phi(m)$ for m = 240

m = 16·15 = $2^4 \cdot 3^1 \cdot 5^1$

= $p_1^{e_1}\, p_2^{e_2}\, p_3^{e_3}$

$$\Phi(240) = \prod_{i=1}^{3}(p_i^{e_i} - p_i^{e_i-1}) = (2^4-2^3)(3^1-3^0)(5^1-5^0)$$

$$= 8 \cdot 2 \cdot 4 = 64$$

e.g. 2) compute $\Phi(m)$ for m = 100

Euler’s Theorem

- Used in public-key cryptography.

Euler’s Theorem:

Let a and m be integers with gcd(a,m) = 1, then:

$$a^{\Phi(m)} \equiv 1 \text{ mod m}$$

- **e.g., Let's check with m = 12 and a = 5.**

$$\Phi(12) = \Phi(2^2 \cdot 3) = (2^2 - 2^1)(3^1 - 3^0) = (4-2)(3-1) = 4$$

$$5^{\Phi(12)} = 5^4 = 25^2 = 625 \equiv 1 \text{ mod } 12$$

Fermat's Little Theorem

➢ Fermat's Little Theorem:

- Given a **prime *p*** and an **integer *a***: $a^p \equiv a \pmod{p}$
- Can be rewritten as $a^{p-1} \equiv 1 \pmod{p}$
- **Use: Find modular inverse**, if p is prime. Rewrite to $a \cdot a^{p-2} \equiv 1 \pmod{p}$
- Comparing with definition of the modular inverse $a \cdot a^{-1} \equiv 1 \bmod m$

→ $a^{-1} \equiv a^{p-2} \pmod{p}$ is the modular inverse modulo a prime *p*

Fermat’s Little Theorem

➢ Fermat’s Little Theorem:

Let a be an integer and p be a prime,

then: $a^p \equiv a \bmod p$

so, $a^{p-1} \equiv 1 \bmod p$

or, $a \cdot a^{p-2} \equiv 1 \bmod p$

so, $a^{-1} \equiv a^{p-2} \bmod p$

✓ **e.g., Let’s check with p = 7 and a = 2**

$$a^{p-2} = 2^5 = 32 \equiv 4 \bmod 7$$

$$2 \cdot 4 \equiv 1 \bmod 7$$

Therefore, $2^{-1} \equiv 4 \bmod 7$

Fermat’s Little Theorem and Euler’s Theorem

- Fermat‘s little theorem = special case of Euler‘s Theorem
- for a prime ***p***: $\Phi(p) = (p^1 - p^0) = p - 1$

  → Fermat: $a^{\Phi(p)} = a^{p-1} \equiv 1 \pmod{p}$

Euler's Phi Function

Notes:

If p is prime, then $\Phi(p) = p - 1$

if p and q are prime, then $\Phi(pq) = \Phi(p) \times \Phi(q)$

How to prove that $\Phi(pq) = \Phi(p) \times \Phi(q)$?

Since Z_n has (pq-1) positive integers.

Since integers that are not relatively prime to n are {p,2p,…(q-1)p} and {q,2q,..,(p-1)q} …
i.e., (p-1) elements + (q-1) elements.

Then the number of integers in Z_n that are relatively prime to n = (pq-1) – [(p-1)+(q-1)]

i.e., pq – (p+q) +1

Then $\Phi(n)$=(p-1)x(q-1) = $\Phi(p) \times \Phi($q$)$

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See You next Lectures!!

Any Question?

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